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Abstract

The aim of this work is to study a class of nonsmooth dynamic contact problem which model several surface interactions, including relaxed unilateral contact conditions, adhesion and Coulomb friction laws, between two viscoelastic bodies of Kelvin-Voigt type. An abstract formulation which generalizes these problems is considered and the existence of a solution is proved by using Ky Fan's fixed point theorem, suitable approximation properties, several estimates and compactness arguments.

1 Introduction

This paper is concerned with the study of a class of dynamic contact problems which describe various surface interactions between two Kelvin-Voigt viscoelastic bodies. These interactions can include some relaxed unilateral contact, pointwise friction or adhesion conditions.

Existence and approximation of solutions to the quasistatic elastic problems have been studied for different contact conditions. The quasistatic unilateral contact problems with local Coulomb friction have been studied in [1, 29, 30] and adhesion laws based on the evolution of intensity of adhesion were analyzed in [28, 10]. The normal compliance models, which are particular regularizations of the Signorini's conditions, have been investigated by several authors, see e.g. [17, 15, 31] and references therein.

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A unified approach, which can be applied to various quasistatic problems, including unilateral and bilateral contact with nonlocal friction, or normal compliance conditions, has been considered recently in [2].

The corresponding dynamic contact problems are more difficult to solve than the quasistatic ones, even in the viscoelastic case. Dynamic frictional contact problems with normal compliance laws for a viscoelastic body have been studied in [22, 17, 18, 5, 24]. Nonlocal friction laws, obtained by suitable regularizations of the normal component of the stress vector appearing in the Coulomb friction conditions, were considered for viscoelastic bodies in [16, 19, 20, 13, 7, 11]. Dynamic frictionless problems with adhesion have been studied in [6, 21, 33] and dynamic viscoelastic problems coupling unilateral contact, recoverable adhesion and nonlocal friction have been analyzed in [12, 9].

Note that, using the Clarke subdifferential, the variational formulations of various contact problems can be given as hemivariational inequalities, which represent a broad generalization of the variational inequalities to locally Lipschitz functions, see [24, 23, 25, 26] and references therein.

A static contact problem with relaxed unilateral conditions and pointwise Coulomb friction was studied in [27], based on new abstract formulations and on Ky Fan's fixed point theorem. The extension to an elastic quasistatic contact problem was investigated in [8].

This work extends the results in [27] to a new class of nonsmooth dynamic contact problems in viscoelasticity, which constitutes a unified approach to study some complex surface interactions.

The paper is organized as follows. In Section 2 the classical formulation of the dynamic contact problem is presented and the variational formulation is given as a two-field problem. Section 3 is devoted to the study of a more general evolution variational inequality, which is written as an equivalent fixed point problem, based on some existence and uniqueness results proved in [11]. Using the Ky Fan's theorem, the existence of a fixed point is proved. In Section 4 this abstract result is used to prove the existence of a variational solution of the dynamic contact problem.

2 Classical and variational formulations

We consider two viscoelastic bodies, characterized by a Kelvin-Voigt constitutive law, which occupy the reference domains Ω^α of $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz boundaries $\Gamma^\alpha = \partial\Omega^\alpha$, $\alpha = 1, 2$.

Let Γ_U^α, Γ_F^α and Γ_C^α be three open disjoint sufficiently smooth parts of Γ^α such that $\Gamma^\alpha = \overline{\Gamma}_U^\alpha \cup \overline{\Gamma}_F^\alpha \cup \overline{\Gamma}_C^\alpha$ and, to simplify the estimates, $\text{meas}(\Gamma_U^\alpha) >$

$0,\ \alpha = 1, 2$. In this paper we assume the small deformation hypothesis and we use Cartesian coordinate representations.

Let $\boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t)$ denote the position at time $t \in [0, T]$, where $0 < T < +\infty$, of the material point represented by $\boldsymbol{x}^\alpha$ in the reference configuration, and $\boldsymbol{u}^\alpha(\boldsymbol{x}^\alpha, t) := \boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t) - \boldsymbol{x}^\alpha$ denote the displacement vector of $\boldsymbol{x}^\alpha$ at time t, with the Cartesian coordinates $u^\alpha = (u_1^\alpha, ..., u_d^\alpha) = (\bar{u}^\alpha, u_d^\alpha)$. Let $\boldsymbol{\varepsilon}^\alpha$, with the Cartesian coordinates $\varepsilon^\alpha = (\varepsilon_{ij}(u^\alpha))$, and $\boldsymbol{\sigma}^\alpha$, with the Cartesian coordinates $\sigma^\alpha = \left(\sigma_{ij}^\alpha\right)$, be the infinitesimal strain tensor and the stress tensor, respectively, corresponding to Ω^α, $\alpha = 1, 2$.

Assume that the displacement $\boldsymbol{U}^\alpha = \boldsymbol{0}$ on $\Gamma_U^\alpha \times (0, T)$, $\alpha = 1, 2$, and that the densities of both bodies are equal to 1. Let $\boldsymbol{f}_1 = (\boldsymbol{f}_1^1, \boldsymbol{f}_1^2)$ and $\boldsymbol{f}_2 = (\boldsymbol{f}_2^1, \boldsymbol{f}_2^2)$ denote the given body forces in $\Omega^1 \cup \Omega^2$ and tractions on $\Gamma_F^1 \cup \Gamma_F^2$, respectively. The initial displacements and velocities of the bodies are denoted by $\boldsymbol{u}_0 = (\boldsymbol{u}_0^1, \boldsymbol{u}_0^2)$, $\boldsymbol{u}_1 = (\boldsymbol{u}_1^1, \boldsymbol{u}_1^2)$. The usual summation convention will be used for $i, j, k, l = 1, \ldots, d$.

Suppose that the solids can be in contact between the potential contact surfaces Γ_C^1 and Γ_C^2 which can be parametrized by two C^1 functions, φ^1, φ^2, defined on an open and bounded subset Ξ of $\mathbb{R}^{d-1}$, such that $\varphi^1(\xi) - \varphi^2(\xi) \geq 0\ \ \forall \xi \in \Xi$ and each Γ_C^α is the graph of φ^α on Ξ that is $\Gamma_C^\alpha = \{\ (\xi,\ \varphi^\alpha(\xi)) \in \mathbb{R}^d\,;\ \xi \in \Xi\}$, $\alpha = 1,\ 2$. Define the initial normalized gap between the two contact surfaces by

$$g_0(\xi) = \frac{\varphi^1(\xi) - \varphi^2(\xi)}{\sqrt{1 + |\nabla\varphi^1(\xi)|^2}} \quad \forall \xi\ \in \Xi.$$

Let $\boldsymbol{n}^\alpha$ denote the unit outward normal vector to Γ^α, $\alpha = 1,\ 2$. We shall use the following notations for the normal and tangential components of a displacement field $\boldsymbol{v}^\alpha$, $\alpha = 1,\ 2$, of the relative displacement corresponding to $\boldsymbol{v} := (\boldsymbol{v}^1, \boldsymbol{v}^2)$ and of the stress vector $\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha$ on Γ_C^α:

$$\begin{aligned}
&\boldsymbol{v}^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, \varphi^\alpha(\xi), t),\ v_N^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, t) \cdot \boldsymbol{n}^\alpha(\xi),\\
&v_N(\xi, t) := v_N^1(\xi, t) + v_N^2(\xi, t),\ [v_N](\xi, t) := v_N(\xi, t) - g_0(\xi),\\
&\boldsymbol{v}_T^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, t) - v_N^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi),\ \boldsymbol{v}_T(\xi, t) := \boldsymbol{v}_T^1(\xi, t) - \boldsymbol{v}_T^2(\xi, t),\\
&\sigma_N^\alpha(\xi, t) := (\boldsymbol{\sigma}^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi)) \cdot \boldsymbol{n}^\alpha(\xi),\ \boldsymbol{\sigma}_T^\alpha(\xi, t) = \boldsymbol{\sigma}^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi) - \sigma_N^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi),
\end{aligned}$$

for all $\xi \in \Xi$ and for all $t \in [0, T]$. Let $g := -[u_N] = g_0 - u_N^1 - u_N^2$ be the gap corresponding to the solution $\boldsymbol{u} := (\boldsymbol{u}^1, \boldsymbol{u}^2)$. Since the displacements, their derivatives and the gap are assumed small, by using a similar method as the one presented in [3] (see also [11]) we obtain the following unilateral contact condition at time t in the set Ξ: $[u_N](\xi, t) = -g(\xi, t) \leq 0\ \ \forall \xi \in \Xi$.

2.1 Classical formulation

Let $\boldsymbol{\mathcal{A}}^\alpha$, $\boldsymbol{\mathcal{B}}^\alpha$ denote two fourth-order tensors, the elasticity tensor and the viscosity tensor corresponding to Ω^α, with the components $\mathcal{A}^\alpha = (\mathcal{A}^\alpha_{ijkl})$ and $\mathcal{B}^\alpha = (\mathcal{B}^\alpha_{ijkl})$, respectively. We assume that these components satisfy the following classical symmetry and ellipticity conditions: $\mathcal{C}^\alpha_{ijkl} = \mathcal{C}^\alpha_{jikl} = \mathcal{C}^\alpha_{klij} \in L^\infty(\Omega^\alpha)$, $\forall\, i,\, j,\, k,\, l = 1, \ldots, d$, $\exists\, \alpha_{\mathcal{C}^\alpha} > 0$ such that $\mathcal{C}^\alpha_{ijkl}\tau_{ij}\tau_{kl} \geq \alpha_{\mathcal{C}^\alpha}\, \tau_{ij}\tau_{ij}$ $\forall\, \tau = (\tau_{ij})$ verifying $\tau_{ij} = \tau_{ji}, \forall\, i,\, j = 1, \ldots, d$, where $\mathcal{C}^\alpha_{ijkl} = \mathcal{A}^\alpha_{ijkl}$, $\mathcal{C}^\alpha = \mathcal{A}^\alpha$ or $\mathcal{C}^\alpha_{ijkl} = \mathcal{B}^\alpha_{ijkl}$, $\mathcal{C}^\alpha = \mathcal{B}^\alpha$ $\forall\, i,\, j,\, k,\, l = 1, \ldots, d$, $\alpha = 1, 2$.

We choose the following state variables: the infinitesimal strain tensor $(\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) = (\boldsymbol{\varepsilon}(\boldsymbol{u}^1), \boldsymbol{\varepsilon}(\boldsymbol{u}^2))$ in $\Omega^1 \cup \Omega^2$, the relative normal displacement $[u_N] = u^1_N + u^2_N - g_0$, and the relative tangential displacement $\boldsymbol{u}_T = \boldsymbol{u}^1_T - \boldsymbol{u}^2_T$ in Ξ.

Let $\underline{\kappa}$, $\overline{\kappa} : \mathbb{R} \to \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\overline{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\underline{\kappa}(s) \leq \overline{\kappa}(s) \text{ and } 0 \notin (\underline{\kappa}(s), \overline{\kappa}(s)) \;\; \forall\, s \in \mathbb{R}, \tag{1}$$

$$\exists\, r_0 \geq 0 \text{ such that } \max(|\underline{\kappa}(s)|, |\overline{\kappa}(s)|) \leq r_0 \;\; \forall\, s \in \mathbb{R}. \tag{2}$$

Consider the following dynamic viscoelastic contact problem with Coulomb friction.

Problem $\boldsymbol{P_c}$: Find $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$ and, for all $t \in (0, T)$,

$$\ddot{\boldsymbol{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{f}^\alpha_1 \text{ in } \Omega^\alpha, \tag{3}$$

$$\boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{\mathcal{A}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + \boldsymbol{\mathcal{B}}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) \text{ in } \Omega^\alpha, \tag{4}$$

$$\boldsymbol{u}^\alpha = \boldsymbol{0} \text{ on } \Gamma^\alpha_U, \;\; \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha = \boldsymbol{f}^\alpha_2 \text{ on } \Gamma^\alpha_F, \;\; \alpha = 1, 2, \tag{5}$$

$$\boldsymbol{\sigma}^1 \boldsymbol{n}^1 + \boldsymbol{\sigma}^2 \boldsymbol{n}^2 = \boldsymbol{0} \text{ in } \Xi, \tag{6}$$

$$\underline{\kappa}([u_N]) \leq \sigma_N \leq \overline{\kappa}([u_N]) \text{ in } \Xi, \tag{7}$$

$$|\boldsymbol{\sigma}_T| \leq \mu\, |\sigma_N| \quad \text{in } \Xi \text{ and} \tag{8}$$

$$|\boldsymbol{\sigma}_T| < \mu\, |\sigma_N| \Rightarrow \dot{\boldsymbol{u}}_T = \boldsymbol{0},$$
$$|\boldsymbol{\sigma}_T| = \mu\, |\sigma_N| \;\Rightarrow \exists\, \tilde{\theta} \geq 0, \;\; \dot{\boldsymbol{u}}_T = -\tilde{\theta} \boldsymbol{\sigma}_T,$$

where $\boldsymbol{\sigma}^\alpha = \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha)$, $\alpha = 1, 2$, $\sigma_N := \sigma^1_N$, $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}^1_T$, and $\mu \in L^\infty(\Xi)$, $\mu \geq 0$ a.e. in Ξ, is the coefficient of friction.

Different choices for $\underline{\kappa}$, $\overline{\kappa}$ will give various contact and friction conditions as can be seen in the following examples.

Example 1. (Adhesion and friction conditions)

Let $s_0 \geq 0$, $M \geq 0$ be constants, $k : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $k \geq 0$ with $k(0) = 0$ and define

$$\underline{\kappa}(s) = \begin{cases} 0 & \text{if } \;\; s \leq -s_0, \\ k(s) & \text{if } \;\; -s_0 < s < 0, \\ -M & \text{if } \;\; s \geq 0, \end{cases} \quad \overline{\kappa}(s) = \begin{cases} 0 & \text{if } \;\; s < -s_0, \\ k(s) & \text{if } \;\; -s_0 \leq s \leq 0, \\ -M & \text{if } \;\; s > 0. \end{cases}$$

Example 2. (Friction condition)

In Example 1 we set $k = s_0 = 0$ and define

$$\underline{\kappa}_M(s) = \begin{cases} 0 & \text{if} \quad s < 0, \\ -M & \text{if} \quad s \geq 0, \end{cases} \quad \overline{\kappa}_M(s) = \begin{cases} 0 & \text{if} \quad s \leq 0, \\ -M & \text{if} \quad s > 0. \end{cases}$$

The classical Signorini's conditions correspond formally to $M = +\infty$.

Example 3. (General normal compliance conditions)

Various normal compliance conditions, friction and adhesion laws can be obtained from the previous general formulation if one considers $\underline{\kappa} = \overline{\kappa} = \kappa$, where $\kappa : \mathbb{R} \to \mathbb{R}$ is some bounded Lipschitz continuous function with $\kappa(0) = 0$, so that σ_N is given by the relation $\sigma_N = \kappa([u_N])$.

2.2 Variational formulations

We adopt the following notations:

$$\begin{aligned} &\boldsymbol{H}^s(\Omega^\alpha) := H^s(\Omega^\alpha; \mathbb{R}^d), \ \alpha = 1, 2, \ \boldsymbol{H}^s := \boldsymbol{H}^s(\Omega^1) \times \boldsymbol{H}^s(\Omega^2), \\ &\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{-s,s} = \langle \boldsymbol{v}^1, \boldsymbol{w}^1 \rangle_{\boldsymbol{H}^{-s}(\Omega^1) \times \boldsymbol{H}^s(\Omega^1)} + \langle \boldsymbol{v}^2, \boldsymbol{w}^2 \rangle_{\boldsymbol{H}^{-s}(\Omega^2) \times \boldsymbol{H}^s(\Omega^2)} \\ &\forall\, \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{H}^{-s}, \forall\, \boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{H}^s, \quad \forall\, s \in \mathbb{R}. \end{aligned}$$

Define the Hilbert spaces $(\boldsymbol{H}, |.|)$ with the associated inner product denoted by $(.\,,.)$, $(\boldsymbol{V}, \|.\|)$ with the associated inner product (of $\boldsymbol{H}^1$) denoted by $\langle .\,,. \rangle$, and the closed convex cones $L^2_+(\Xi)$, $L^2_+(\Xi \times (0,T))$ as follows:

$$\begin{aligned} &\boldsymbol{H} := \boldsymbol{H}^0 = L^2(\Omega^1; \mathbb{R}^d) \times L^2(\Omega^2; \mathbb{R}^d), \ \boldsymbol{V} := \boldsymbol{V}^1 \times \boldsymbol{V}^2, \ \text{ where} \\ &\boldsymbol{V}^\alpha = \{\boldsymbol{v}^\alpha \in \boldsymbol{H}^1(\Omega^\alpha);\, \boldsymbol{v}^\alpha = \boldsymbol{0} \ \text{ a.e. on } \Gamma^\alpha_U\}, \ \alpha = 1, 2, \\ &L^2_+(\Xi) := \{\delta \in L^2(\Xi);\, \delta \geq 0 \ \text{ a.e. in } \ \Xi\}, \\ &L^2_+(\Xi \times (0,T)) := \{\eta \in L^2(0,T; L^2(\Xi));\, \eta \geq 0 \ \text{ a.e. in } \ \Xi \times (0,T)\}. \end{aligned}$$

Let a, b be two bilinear, continuous and symmetric mappings defined on $\boldsymbol{H}^1 \times \boldsymbol{H}^1 \to \mathbb{R}$ by

$$a(\boldsymbol{v}, \boldsymbol{w}) = a^1(\boldsymbol{v}^1, \boldsymbol{w}^1) + a^2(\boldsymbol{v}^2, \boldsymbol{w}^2), \ b(\boldsymbol{v}, \boldsymbol{w}) = b^1(\boldsymbol{v}^1, \boldsymbol{w}^1) + b^2(\boldsymbol{v}^2, \boldsymbol{w}^2)$$

$$\forall\, \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2), \ \boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{H}^1, \text{ where, for } \ \alpha = 1, 2,$$

$$a^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha) = \int_{\Omega^\alpha} \boldsymbol{\mathcal{A}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\, dx, \ b^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha) = \int_{\Omega^\alpha} \boldsymbol{\mathcal{B}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\, dx.$$

Assume $\boldsymbol{f}_1^\alpha \in W^{1,\infty}(0,T;L^2(\Omega^\alpha;\mathbb{R}^d))$, $\boldsymbol{f}_2^\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_F^\alpha;\mathbb{R}^d))$, $\alpha = 1,2$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in \boldsymbol{V}$, $g_0 \in L^2_+(\Xi)$, and define the following mappings:

$$J : L^2(\Xi) \times \boldsymbol{H}^1 \to \mathbb{R},\ J(\delta,\boldsymbol{v}) = \int_\Xi \mu\,|\delta|\,|\boldsymbol{v}_T|\,d\xi$$
$$\forall\,\delta \in L^2(\Xi),\ \forall\,\boldsymbol{v} = (\boldsymbol{v}^1,\boldsymbol{v}^2) \in \boldsymbol{H}^1,$$

$$\boldsymbol{f} \in W^{1,\infty}(0,T;\boldsymbol{H}^1),\ \langle \boldsymbol{f},\boldsymbol{v}\rangle = \textstyle\sum_{\alpha=1,2} \displaystyle\int_{\Omega^\alpha} \boldsymbol{f}_1^\alpha \cdot \boldsymbol{v}^\alpha\,dx + \sum_{\alpha=1,2} \int_{\Gamma_F^\alpha} \boldsymbol{f}_2^\alpha \cdot \boldsymbol{v}^\alpha\,ds$$
$$\forall\,\boldsymbol{v} = (\boldsymbol{v}^1,\boldsymbol{v}^2) \in \boldsymbol{H}^1,\ \forall\,t \in [0,T].$$

Assume the following compatibility conditions: $[u_{0N}] \le 0$, $\overline{\kappa}([u_{0N}]) = 0$ a.e. in Ξ and $\exists\,\boldsymbol{p}_0 \in \boldsymbol{H}$ such that

$$(\boldsymbol{p}_0,\boldsymbol{w}) + a(\boldsymbol{u}_0,\boldsymbol{w}) + b(\boldsymbol{u}_1,\boldsymbol{w}) = \langle \boldsymbol{f}(0),\boldsymbol{w}\rangle \quad \forall\,\boldsymbol{w} \in \boldsymbol{V}. \tag{9}$$

The following compactness theorem proved in [32] will be used several times in this paper.

Theorem 2.1. *Let $\hat{X}$, $\hat{U}$ and $\hat{Y}$ be three Banach spaces such that $\hat{X} \subset \hat{U} \subset \hat{Y}$ with compact embedding from $\hat{X}$ into $\hat{U}$.*

(i) Let $\mathcal{F}$ be bounded in $L^p(0,T;\hat{X})$, where $1 \le p < \infty$, and $\partial\mathcal{F}/\partial t := \{\dot{f};\ f \in \mathcal{F}\}$ be bounded in $L^1(0,T;\hat{Y})$. Then $\mathcal{F}$ is relatively compact in $L^p(0,T;\hat{U})$.

(ii) Let $\mathcal{F}$ be bounded in $L^\infty(0,T;\hat{X})$ and $\partial\mathcal{F}/\partial t$ be bounded in $L^r(0,T;\hat{Y})$, where $r > 1$. Then $\mathcal{F}$ is relatively compact in $C([0,T];\hat{U})$.

For every $\zeta \in L^2(0,T;L^2(\Xi)) = L^2(\Xi \times (0,T))$, define the following sets:

$$\Lambda(\zeta) = \{\eta \in L^2(0,T;L^2(\Xi)); \underline{\kappa} \circ \zeta \le \eta \le \overline{\kappa} \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$
$$\Lambda_+(\zeta) = \{\eta \in L^2_+(\Xi \times (0,T)); \underline{\kappa}_+ \circ \zeta \le \eta \le \overline{\kappa}_+ \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$
$$\Lambda_-(\zeta) = \{\eta \in L^2_+(\Xi \times (0,T)); \overline{\kappa}_- \circ \zeta \le \eta \le \underline{\kappa}_- \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$

where, for each $r \in \mathbb{R}$, $r_+ := \max(0,r)$ and $r_- := \max(0,-r)$ denote the positive and negative parts, respectively.

For each $\zeta \in L^2(0,T;L^2(\Xi))$ the sets $\Lambda(\zeta)$, $\Lambda_+(\zeta)$ and $\Lambda_-(\zeta)$ are clearly nonempty, because the bounding functions belong to the respective set, closed and convex.

Since meas$(\Xi) < \infty$ and $\underline{\kappa}$, $\overline{\kappa}$ satisfy (2), it is also readily seen that there exists a constant, denoted by R_0 and depending on meas(Ξ), r_0 and T, such that for all $\zeta \in L^2(0,T;L^2(\Xi))$ the sets $\Lambda_+(\zeta)$ and $\Lambda_-(\zeta)$ are bounded in norm in $L^2(0,T;L^2(\Xi))$ by R_0. Moreover, these sets are bounded in $L^\infty(0,T;L^\infty(\Xi))$.

A first variational formulation of the problem P_c is the following.
Problem $\boldsymbol{P_v^1}$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, $\lambda \in \Lambda([u_N])$ and

$$
\begin{aligned}
&\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T) \rangle_{-\iota,\,\iota} - (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0) - \int_0^T (\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}})\, dt \\
&\quad + \int_0^T \Big\{ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}) - (\lambda, v_N - u_N)_{L^2(\Xi)} \Big\}\, dt \qquad (10)\\
&\quad + \int_0^T \{ J(\lambda, \boldsymbol{v} + k\dot{\boldsymbol{u}} - \boldsymbol{u}) - J(\lambda, k\dot{\boldsymbol{u}}) \}\, dt \geq \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle\, dt \\
&\quad \forall \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}) \cap W^{1,2}(0,T;\boldsymbol{H}), \quad \text{where} \ \ 1 > \iota > \frac{1}{2}, \ \ k > 0.
\end{aligned}
$$

The formal equivalence between the variational problem P_v^1 and the classical problem (3)–(8) can be easily proved by using Green's formula and an integration by parts, where the Lagrange multiplier λ satisfies the relation $\lambda = \sigma_N$.

Let $\phi : L^2_+(\Xi) \times L^2_+(\Xi) \times \boldsymbol{V} \to \mathbb{R}$ be defined by

$$
\begin{aligned}
&\phi(\delta_1, \delta_2, \boldsymbol{v}) = -(\delta_1 - \delta_2, v_N)_{L^2(\Xi)} + \int_\Xi \mu\, (\delta_1 + \delta_2)\, |\boldsymbol{v}_T|\, d\xi \\
&\qquad \forall\, (\delta_1, \delta_2) \in (L^2_+(\Xi))^2, \ \forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{V}.
\end{aligned} \qquad (11)
$$

Since $\eta \in \Lambda(\zeta)$ if and only if $(\eta_+, \eta_-) \in \Lambda_+(\zeta) \times \Lambda_-(\zeta)$, it follows that the variational problem P_v^1 is clearly equivalent with the following problem.
Problem $\boldsymbol{P_v^2}$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, $(\lambda_+, \lambda_-) \in \Lambda_+([u_N]) \times \Lambda_-([u_N])$ and

$$
\begin{aligned}
&\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T) \rangle_{-\iota,\,\iota} - (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0) \\
&\quad + \int_0^T \{ -(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}) + a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}) \}\, dt \qquad (12)\\
&\quad + \int_0^T \{ \phi(\lambda_+, \lambda_-, \boldsymbol{v} + k\dot{\boldsymbol{u}} - \boldsymbol{u}) - \phi(\lambda_+, \lambda_-, k\dot{\boldsymbol{u}}) \}\, dt \geq \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle\, dt \\
&\quad \forall \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}) \cap W^{1,2}(0,T;\boldsymbol{H}).
\end{aligned}
$$

The existence of variational solutions to problem P_c will be established by using some abstract existence results that will be presented in the following section.

3 General existence results

Let U_0, $(V_0,\, \|.\|,\, \langle\, .\, ,.\rangle)$, $(U,\, \|.\|_U)$ and $(H_0,\, |.|,\, (\, .\, ,.))$ be four Hilbert spaces such that U_0 is a closed linear subspace of V_0 dense in H_0, $V_0 \subset U \subseteq H_0$ with continuous embeddings and the embedding from V_0 into U is compact.

To simplify the presentation and in view of the applications to contact problems, $L^2(\Xi)$ will be preserved in the abstract formulation even if, more generally, the space $L^2(\hat{\Xi})$ can be considered with $(\hat{\Xi}, m)$ a finite and complete measure space, see [27] for a time-independent application. Also, we use the notation $\Xi_T := \Xi \times (0,T)$.

Let $a_0,\, b_0 : V_0 \times V_0 \to \mathbb{R}$ be two bilinear and symmetric forms such that

$$\exists\, M_a,\, M_b > 0 \quad a_0(u,v) \le M_a \,\|u\|\,\|v\|, \quad b_0(u,v) \le M_b \,\|u\|\,\|v\|, \tag{13}$$

$$\exists\, m_a,\, m_b > 0 \quad a_0(v,v) \ge m_a \,\|v\|^2, \quad b_0(v,v) \ge m_b \,\|v\|^2 \quad \forall\, u, v \in V_0. \tag{14}$$

Let $l : V_0 \to L^2(\Xi)$ and $\phi_0 : L^2_+(\Xi) \times L^2_+(\Xi) \times V_0 \to \mathbb{R}$ be two mappings satisfying the following conditions:

$$\begin{aligned} &\exists\, k_1 > 0 \text{ such that } \forall\, v_1, v_2 \in V_0, \\ &\|l(v_1) - l(v_2)\|_{L^2(\Xi)} \le k_1 \|v_1 - v_2\|_U, \end{aligned} \tag{15}$$

$$\forall\, \gamma_1,\, \gamma_2 \in L^2_+(\Xi),\ \forall\, \theta \ge 0,\ \forall\, v_1,\, v_2,\, v \in V_0,$$

$$\phi_0(\gamma_1, \gamma_2, v_1 + v_2) \le \phi_0(\gamma_1, \gamma_2, v_1) + \phi_0(\gamma_1, \gamma_2, v_2), \tag{16}$$

$$\phi_0(\gamma_1, \gamma_2, \theta v) = \theta\, \phi_0(\gamma_1, \gamma_2, v), \tag{17}$$

$$\forall\, v \in V_0, \quad \phi_0(0, 0, v) = 0, \tag{18}$$

$$\forall\, \gamma_1,\, \gamma_2 \in L^2_+(\Xi),\ \forall\, v \in U_0, \quad \phi_0(\gamma_1, \gamma_2, v) = 0, \tag{19}$$

$$\begin{aligned} &\exists\, k_2 > 0 \text{ such that } \forall\, \gamma_1,\, \gamma_2,\, \delta_1,\, \delta_2 \in L^2_+(\Xi), \forall\, v_1, v_2 \in V_0, \\ &|\phi_0(\gamma_1, \gamma_2, v_1) - \phi_0(\gamma_1, \gamma_2, v_2) + \phi_0(\delta_1, \delta_2, v_2) - \phi_0(\delta_1, \delta_2, v_1)| \\ &\quad \le k_2(\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\|v_1 - v_2\|_U, \end{aligned} \tag{20}$$

$$\begin{aligned} &\text{if } (\gamma_1^n, \gamma_2^n) \in (L^2_+(\Xi_T))^2 \text{ for all } n \in \mathbb{N} \\ &\text{and } (\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2) \text{ in } (L^2(0,T; L^2(\Xi)))^2, \text{ then} \\ &\int_0^T \phi_0(\gamma_1^n, \gamma_2^n, v)\, ds \to \int_0^T \phi_0(\gamma_1, \gamma_2, v)\, ds \quad \forall\, v \in L^2(0,T; V_0). \end{aligned} \tag{21}$$

Remark 3.1. *i) Since by (17) $\phi_0(\cdot, \cdot, 0) = 0$, from (20), for $v_2 = 0$, $v_1 = v$, we have*

$$\begin{aligned} &\forall\, \gamma_1,\, \gamma_2,\, \delta_1,\, \delta_2 \in L^2_+(\Xi), \forall\, v \in V_0, \\ &|\phi_0(\gamma_1, \gamma_2, v) - \phi_0(\delta_1, \delta_2, v)| \le k_2(\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\|v\|_U. \end{aligned} \tag{22}$$

ii) From (18) and (20), for $\delta_1 = \delta_2 = 0$, we derive

$$\begin{array}{l}\forall\, \gamma_1,\ \gamma_2\ \in L^2_+(\Xi),\ \forall\, v_1, v_2 \in V_0,\\ |\phi_0(\gamma_1,\gamma_2,v_1) - \phi_0(\gamma_1,\gamma_2,v_2)| \le k_2(\|\gamma_1\|_{L^2(\Xi)} + \|\gamma_2\|_{L^2(\Xi)})\|v_1 - v_2\|_U.\end{array} \tag{23}$$

iii) If $(\gamma_1^n, \gamma_2^n) \in (L^2_+(\Xi_T))^2$, for all $n \in \mathbb{N}$, $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0,T;L^2(\Xi)))^2$, and $v_m \to v$ in $L^2(0,T;U)$, then

$$\lim_{n,m\to\infty} \int_0^T \phi_0(\gamma_1^n, \gamma_2^n, v_m)\, ds \to \int_0^T \phi_0(\gamma_1,\gamma_2,v)\, ds, \tag{24}$$

which can be proved by taking into account (23) in the following relations:

$$\begin{array}{l} \displaystyle |\int_0^T \{\phi_0(\gamma_1^n,\gamma_2^n,v_m) - \phi_0(\gamma_1,\gamma_2,v)\}\, ds| \\ \displaystyle \le \int_0^T |\phi_0(\gamma_1^n,\gamma_2^n,v_m) - \phi_0(\gamma_1^n,\gamma_2^n,v)|\, ds + |\int_0^T \{\phi_0(\gamma_1^n,\gamma_2^n,v) - \phi_0(\gamma_1,\gamma_2,v)\}\, ds| \\ \displaystyle \le \int_0^T k_2(\|\gamma_1^n\|_{L^2(\Xi)} + \|\gamma_2^n\|_{L^2(\Xi)})\|v_m - v\|_U\, ds \\ \displaystyle +|\int_0^T \{\phi_0(\gamma_1^n,\gamma_2^n,v) - \phi_0(\gamma_1,\gamma_2,v)\}\, ds|, \end{array}$$

and passing to limits by using (21) and that $(\gamma_{1,2}^n)_n$ are bounded in $L^2(0,T;L^2(\Xi))$.

Assume that $f_0 \in W^{1,\infty}(0,T;V_0)$, u^0, $u^1 \in V_0$ are given, and that the following compatibility condition holds: $\overline{\kappa}(l(u^0)) = 0$ and $\exists\, p_0 \in H_0$ such that

$$(p_0, w) + a_0(u^0, w) + b_0(u^1, w) = \langle f_0(0), w\rangle \quad \forall\, w \in V_0. \tag{25}$$

Consider the following problem.
Problem $\boldsymbol{Q_1}$: Find $u \in W_0$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $u(0) = u^0$, $\dot{u}(0) = u^1$, $(\lambda_+, \lambda_-) \in \Lambda_+(l(u)) \times \Lambda_-(l(u))$ and

$$\begin{array}{l} \langle \dot{u}(T), v(T) - u(T)\rangle_{U'\times U} - (u_1, v(0) - u_0) \\ \displaystyle + \int_0^T \{-(\dot{u}, \dot{v} - \dot{u}) + a_0(u, v-u) + b_0(\dot{u}, v-u)\}\, dt \qquad (26) \\ \displaystyle + \int_0^T \{\phi_0(\lambda_+, \lambda_-, v + k\dot{u} - u) - \phi_0(\lambda_+,\lambda_-, k\dot{u})\}\, dt \ge \int_0^T \langle f_0, v-u\rangle\, dt \\ \forall\, v\ \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0), \end{array}$$

where $W_0 := C^1([0,T];U') \cap W^{1,2}(0,T;V_0)$.

The sets $\Lambda_+(\zeta)$, $\Lambda_-(\zeta)$ and $\Lambda(\zeta)$ have the following useful properties, see also [27].

Lemma 3.1. *Let $\zeta \in L^2(0,T;L^2(\Xi))$ and $(\eta_1,\eta_2) \in \Lambda_+(\zeta)\times\Lambda_-(\zeta)$. Then $\eta_1\eta_2 = 0$ a.e. in Ξ_T and there exists $\eta \in \Lambda(\zeta)$ such that $\eta_+ = \eta_1$, $\eta_- = \eta_2$ a.e. in Ξ_T.*

Proof. If $\underline{\kappa}_+ \circ \zeta \le \eta_1 \le \overline{\kappa}_+ \circ \zeta$ and $\overline{\kappa}_- \circ \zeta \le \eta_2 \le \underline{\kappa}_- \circ \zeta$ a.e. in Ξ_T, then

$$(\underline{\kappa}_+ \circ \zeta)\,(\overline{\kappa}_- \circ \zeta) \le \eta_1\eta_2 \le (\overline{\kappa}_+ \circ \zeta)\,(\underline{\kappa}_- \circ \zeta) \text{ a.e. in } \Xi_T. \tag{27}$$

Since by (1) $0 \notin (\underline{\kappa}(\zeta(\xi,t)),\overline{\kappa}(\zeta(\xi,t)))$, it follows that for almost all $(\xi,t)\in\Xi_T$ the terms $\underline{\kappa}(\zeta(\xi,t))$ and $\overline{\kappa}(\zeta(\xi,t))$ have the same sign, or at least one term is equal to zero. Thus, $(\underline{\kappa}_+ \circ \zeta)\,(\overline{\kappa}_- \circ \zeta) = (\overline{\kappa}_+ \circ \zeta)\,(\underline{\kappa}_- \circ \zeta) = 0$ a.e. in Ξ_T, so that by (27) one obtains $\eta_1\eta_2 = 0$ a.e. in Ξ_T.

To complete the proof, it suffices to take $\eta = \eta_1 - \eta_2$ and, using the relations $\eta_1 \ge 0$, $\eta_2 \ge 0$ and $\eta_1\,\eta_2 = 0$ a.e. in Ξ_T, to see that $\eta_+ = \eta_1$, $\eta_- = \eta_2$ a.e. in Ξ_T. □

Based on the previous lemma, consider the following problem, which has the same solution u as the problem Q_1, and the solutions λ_1, λ_2 satisfy the relation $\lambda = \lambda_1 - \lambda_2$, where λ is a solution of Q_1 .
Problem $\boldsymbol{Q_2}$: Find $u \in W_0$, $\lambda_1, \lambda_2 \in L^2(0,T;L^2(\Xi))$ such that $u(0)=u^0$, $\dot u(0) = u^1$, $(\lambda_1,\lambda_2)\in\Lambda_+(l(u))\times\Lambda_-(l(u))$ and

$$\begin{aligned}
&\langle \dot u(T), v(T)-u(T)\rangle_{U'\times U} - (u_1, v(0)-u_0)\\
&+\int_0^T \{-(\dot u,\dot v-\dot u) + a_0(u,v-u) + b_0(\dot u, v-u)\}\,dt \qquad (28)\\
&+\int_0^T \{\phi_0(\lambda_1,\lambda_2,v+k\dot u-u) - \phi_0(\lambda_1,\lambda_2,k\dot u)\}\,dt \ge \int_0^T \langle f_0, v-u\rangle\,dt\\
&\forall\, v \in L^\infty(0,T;V_0)\cap W^{1,2}(0,T;H_0).
\end{aligned}$$

3.1 Some auxiliary existence results

For the convenience of the reader, an existence and uniqueness result proved in [11] will be restated here, under an adapted form.

Let $\beta : V_0 \to \mathbb{R}$ and $\phi_1 : [0,T]\times V_0^3 \to \mathbb{R}$ be two sequentially weakly continuous mappings such that

$$\beta(0)=0 \text{ and } \phi_1(t,z,v,w_1+w_2)\le \phi_1(t,z,v,w_1)+\phi_1(t,z,v,w_2), \tag{29}$$

$$\phi_1(t,z,v,\theta w)=\theta\,\phi_1(t,z,v,w), \tag{30}$$

$$\phi_1(0,0,0,w)=0 \;\; \forall\, t\in[0,T],\ \forall\, z,\ v,\ w,\ w_{1,2}\in V_0,\ \forall\,\theta\ge 0, \tag{31}$$

$$\begin{gathered}\exists\, k_3>0 \text{ such that } \forall\, t_{1,2}\in[0,T],\ \forall\, u_{1,2},v_{1,2},w\in V_0,\\ |\phi_1(t_1,u_1,v_1,w)-\phi_1(t_2,u_2,v_2,w)|\\ \le k_3(|t_1-t_2|+|\beta(u_1-u_2)|+|v_1-v_2|)\,\|w\|,\end{gathered} \tag{32}$$

$$\begin{gathered}\exists\, k_4>0 \text{ such that } \forall\, t_{1,2}\in[0,T],\ \forall\, u_{1,2},v_{1,2},w_{1,2}\in V_0,\\ |\phi_1(t_1,u_1,v_1,w_1)-\phi_1(t_1,u_1,v_1,w_2)+\phi_1(t_2,u_2,v_2,w_2)\\ -\phi_1(t_2,u_2,v_2,w_1)|\le k_4\,(\,|t_1-t_2|+\|u_1-u_2\|+|v_1-v_2|)\,\|w_1-w_2\|.\end{gathered} \tag{33}$$

Let $L\in W^{1,\infty}(0,T;V_0)$ and assume the following compatibility condition on the initial data: $\exists\, p_1\in H_0$ such that

$$(p_1,w)+a_0(u^0,w)+b_0(u^1,w)+\phi_1(0,u^0,u^1,w)=\langle L(0),w\rangle \;\; \forall\, w\in V_0. \tag{34}$$

Consider the following problem.
Problem $\boldsymbol{Q_3}$: Find $u\in W^{2,2}(0,T;H_0)\cap W^{1,2}(0,T;V_0)$ such that $u(0)=u^0$, $\dot u(0)=u^1$, and for almost all $t\in(0,T)$

$$\begin{aligned}&(\ddot u,v-\dot u)+a_0(u,v-\dot u)+b_0(\dot u,v-\dot u)\\ &\quad+\phi_1(t,u,\dot u,v)-\phi_1(t,u,\dot u,\dot u)\ge\langle L,v-\dot u\rangle \quad \forall\, v\in V_0.\end{aligned} \tag{35}$$

Under the assumptions (13), (14), (29), (30), (32)-(34) and the stronger condition

$$\phi_1(t,0,0,w)=0 \quad \forall\, t\in[0,T],\ \forall\, w\in V_0, \tag{36}$$

an existence and uniqueness result for the problem Q_3 was proved in [11] but in its proof the relation (36) was only used to verify that the relation (33) implies that $\phi_1(t,z,v,\cdot)$ is Lipschitz continuous on V_0. Since (31) and (33) also imply that $\phi_1(t,z,v,\cdot)$ is Lipschitz continuous, we clearly have the following existence and uniqueness result.

Theorem 3.1. *Under the assumptions (13), (14), (29)-(34), there exists a unique solution to the problem Q_3.*

Lemma 3.2. *Assume that (13), (14), (16)-(18), (20), and (25) hold. Then, for each $(\gamma_1,\gamma_2)\in(W^{1,\infty}(0,T;L^2(\Xi)))^2\cap(L^2_+(\Xi_T))^2$ with $\gamma_1(0)=\gamma_2(0)=0$, there exists a unique solution $u=u_{(\gamma_1,\gamma_2)}$ of the following evolution variational inequality: find $u\in W^{2,2}(0,T;H_0)\cap W^{1,2}(0,T;V_0)$ such that $u(0)=u^0$,*

$\dot{u}(0) = u^1$, *and for almost all* $t \in (0,T)$

$$\begin{aligned}&(\ddot{u}, v - \dot{u}) + a_0(u, v - \dot{u}) + b_0(\dot{u}, v - \dot{u})\\&\quad +\phi_0(\gamma_1, \gamma_2, v) - \phi_0(\gamma_1, \gamma_2, \dot{u}) \geq \langle f_0, v - \dot{u}\rangle \quad \forall\, v \in V_0.\end{aligned} \tag{37}$$

Proof. We apply Theorem 3.1 to $\beta = 0$, $L = f_0$ and

$$\phi_1(t, z, v, w) = \phi_0(\gamma_1(t), \gamma_2(t), w) \quad \forall\, t \in [0,T],\ \forall\, z,\ v,\ w \in V_0.$$

Since ϕ_0 satisfies (16)-(18) one can easily verify the properties (29)-(31).
Also, (25) and (31) imply the condition (34).
Using (20), we have

$$\begin{aligned}&\forall\, t_{1,2} \in [0,T],\ \forall\, u_{1,2},\ v_{1,2},\ w_{1,2} \in V_0,\\&|\phi_1(t_1, u_1, v_1, w_1) - \phi_1(t_1, u_1, v_1, w_2) + \phi_1(t_2, u_2, v_2, w_2) - \phi_1(t_2, u_2, v_2, w_1)|\\&= |\phi_0(\gamma_1(t_1), \gamma_2(t_1), w_1) - \phi_0(\gamma_1(t_1), \gamma_2(t_1), w_2)|\\&\quad +\phi_0(\gamma_1(t_2), \gamma_2(t_2), w_2) - \phi_0(\gamma_1(t_2), \gamma_2(t_2), w_1)|\\&\leq k_2(\|\gamma_1(t_1) - \gamma_1(t_2)\|_{L^2(\Xi)} + \|\gamma_2(t_1) - \gamma_2(t_2)\|_{L^2(\Xi)})\|w_1 - w_2\|_U\\&\leq k_2(C_{\gamma_1} + C_{\gamma_2})|t_1 - t_2|\|w_1 - w_2\|_U\\&\leq k_5|t_1 - t_2|\|w_1 - w_2\|_U,\end{aligned}$$

where C_{γ_1}, C_{γ_2} denote the Lipschitz constants of γ_1, γ_2, respectively, and $k_5 = k_2(C_{\gamma_1} + C_{\gamma_2})$.

Thus,

$$\begin{aligned}&|\phi_1(t_1, u_1, v_1, w_1) - \phi_1(t_1, u_1, v_1, w_2) + \phi_1(t_2, u_2, v_2, w_2) - \phi_1(t_2, u_2, v_2, w_1)|\\&\leq k_5|t_1 - t_2|\|w_1 - w_2\|_U \quad \forall\, t_{1,2} \in [0,T],\ \forall\, u_{1,2},\ v_{1,2},\ w_{1,2} \in V_0,\end{aligned} \tag{38}$$

and, since by the continuous embedding $V_0 \subset U$ there exists $C_U > 0$ such that $\|w\|_U \leq C_U\|w\| \ \ \forall\, w \in V_0$, it follows that ϕ_1 satisfies (33) with $k_4 = k_5 C_U$.

Taking in (38) $w_1 = w$, $w_2 = 0$, by (30) with $\theta = 0$, we obtain

$$\begin{aligned}&|\phi_1(t_1, u_1, v_1, w) - \phi_1(t_2, u_2, v_2, w)| \leq k_5|t_1 - t_2|\|w\|_U\\&\qquad \forall\, t_{1,2} \in [0,T],\ \forall\, u_{1,2},\ v_{1,2},\ w \in V_0,\end{aligned} \tag{39}$$

and using the continuous embedding $V_0 \subset U$, it follows that ϕ_1 satisfies (32) with $k_3 = k_5 C_U$.

Now, taking in (38) $t_1 = t$, $t_2 = 0$, $u_1 = z$, $v_1 = v$, $u_2 = v_2 = 0$, by (31) we have

$$\begin{gathered} |\phi_1(t, z, v, w_1) - \phi_1(t, z, v, w_2)| \le k_5\, t\, \|w_1 - w_2\|_U \\ \forall\, t \in [0, T],\ \forall\, z,\ v,\ w_{1,2} \in V_0. \end{gathered} \tag{40}$$

As the embedding from V_0 into U is compact, from (39) and (40) it is easily seen that ϕ_1, which is depending only on t and w, is weakly sequentially continuous.

By Theorem 3.1 there exists a unique solution $u = u_{(\gamma_1,\gamma_2)}$ of the variational inequality (37). □

Lemma 3.3. *Let* (γ_1, γ_2), $(\delta_1, \delta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2$ *such that* $\gamma_1(0) = \gamma_2(0) = \delta_1(0) = \delta_2(0) = 0$ *and let* $u_{(\gamma_1,\gamma_2)}$, $u_{(\delta_1,\delta_2)}$ *be the corresponding solutions of (37). Then there exists a constant* $C_0 > 0$, *independent of* (γ_1, γ_2), (δ_1, δ_2), *such that for all* $t \in [0, T]$

$$\begin{aligned} &|\dot u_{(\gamma_1,\gamma_2)}(t) - \dot u_{(\delta_1,\delta_2)}(t)|^2 + \|u_{(\gamma_1,\gamma_2)}(t) - u_{(\delta_1,\delta_2)}(t)\|^2 \\ &\qquad + \int_0^t \|\dot u_{(\gamma_1,\gamma_2)} - \dot u_{(\delta_1,\delta_2)}\|^2\, ds \\ &\le C_0 \int_0^t \{\phi_0(\gamma_1, \gamma_2, \dot u_{(\delta_1,\delta_2)}) - \phi_0(\gamma_1, \gamma_2, \dot u_{(\gamma_1,\gamma_2)}) \\ &\qquad + \phi_0(\delta_1, \delta_2, \dot u_{(\gamma_1,\gamma_2)}) - \phi_0(\delta_1, \delta_2, \dot u_{(\delta_1,\delta_2)})\}\, ds. \end{aligned} \tag{41}$$

Proof. Let (γ_1, γ_2), $(\delta_1, \delta_2) \in (W^{1,\infty}(0, T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2$ and $u_1 := u_{(\gamma_1,\gamma_2)}$, $u_2 := u_{(\delta_1,\delta_2)}$ be the corresponding solutions of (37), for which the existence and uniqueness are insured by Lemma 3.2. Taking in each inequality $v = \dot u_2$ and $v = \dot u_1$, respectively, for a.e. $s \in (0, T)$ we have

$$\begin{aligned} &(\ddot u_1 - \ddot u_2, \dot u_1 - \dot u_2) + a_0(u_1 - u_2, \dot u_1 - \dot u_2) + b_0(\dot u_1 - \dot u_2, \dot u_1 - \dot u_2) \\ &\le \phi_0(\gamma_1, \gamma_2, \dot u_2) - \phi_0(\gamma_1, \gamma_2, \dot u_1) + \phi_0(\delta_1, \delta_2, \dot u_1) - \phi_0(\delta_1, \delta_2, \dot u_2). \end{aligned}$$

As the solutions u_1, u_2 verify the same initial conditions and a_0 is symmetric, by integrating over $(0, t)$ it follows that for all $t \in [0, T]$

$$\begin{aligned} &\frac{1}{2}|\dot u_1(t) - \dot u_2(t)|^2 + \frac{1}{2} a_0(u_1(t) - u_2(t), u_1(t) - u_2(t)) \\ &\quad + \int_0^t b_0(\dot u_1 - \dot u_2, \dot u_1 - \dot u_2)\, ds \\ &\le \int_0^t \{\phi_0(\gamma_1, \gamma_2, \dot u_2) - \phi_0(\gamma_1, \gamma_2, \dot u_1) + \phi_0(\delta_1, \delta_2, \dot u_1) - \phi_0(\delta_1, \delta_2, \dot u_2)\}\, ds. \end{aligned}$$

Using the V_0 - ellipticity conditions (14), the estimate (41) follows. □

3.2 A fixed point problem formulation

Since $\mathcal{D}(0,T;L^2(\Xi))$ is dense in $L^2(0,T;L^2(\Xi))$, which is classically proved by using the convolution product with suitable mollifiers, it follows that for every $\gamma \in L^2_+(\Xi_T)$, there exists a sequence $(\gamma^n)_n$ in $W^{1,\infty}(0,T;L^2(\Xi)) \cap L^2_+(\Xi_T)$ such that $\gamma^n(0) = 0$, for all $n \in \mathbb{N}$, and $\gamma^n \to \gamma$ in $L^2(0,T;L^2(\Xi))$.

Theorem 3.2. *Assume that (13), (14), (16)-(21), and (25) hold. For each $(\gamma_1, \gamma_2) \in (L^2_+(\Xi_T))^2$, let $(\gamma_1^n, \gamma_2^n)_n$ be a sequence in $(W^{1,\infty}(0,T;L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2$ such that $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0,T;L^2(\Xi)))^2$, $\gamma_1^n(0) = \gamma_2^n(0) = 0$, and let $u_{(\gamma_1^n,\gamma_2^n)}$ be the solution of (37) corresponding to (γ_1^n, γ_2^n), for every $n \in \mathbb{N}$. Then $(u_{(\gamma_1^n,\gamma_2^n)})_n$ is strongly convergent in W_0, its limit, denoted by $u = u_{(\gamma_1,\gamma_2)}$, is independent of the chosen sequence converging to (γ_1, γ_2) with the same properties as $(\gamma_1^n, \gamma_2^n)_n$ and is a solution of the following evolution variational inequality: $u(0) = u^0$, $\dot{u}(0) = u^1$,*

$$
\begin{aligned}
&\langle \dot{u}(T), v(T) - u(T)\rangle_{U' \times U} - (u^1, v(0) - u^0) \\
&\quad + \int_0^T \{-(\dot{u}, \dot{v} - \dot{u}) + a_0(u, v - u) + b_0(\dot{u}, v - u)\}\, dt \\
&+ \int_0^T \{\phi_0(\gamma_1, \gamma_2, v - u + k\dot{u}) - \phi_0(\gamma_1, \gamma_2, k\dot{u})\}\, dt \geq \int_0^T \langle f_0, v - u\rangle\, dt \\
&\qquad \forall\, v \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0).
\end{aligned}
\tag{42}
$$

Proof. Assume $(\gamma_1, \gamma_2) \in (L^2_+(\Xi_T))^2$, γ_1^n, $\gamma_2^n \in W^{1,\infty}(0,T;L^2(\Xi)) \cap L^2_+(\Xi_T)$ such that $\gamma_1^n(0) = \gamma_2^n(0) = 0$, for all $n \in \mathbb{N}$ and $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0,T;L^2(\Xi)))^2$. Then, by Lemma 3.2, for every $n \in \mathbb{N}$ there exists a unique solution of the following variational inequality: find $u_n := u_{(\gamma_1^n,\gamma_2^n)} \in W^{2,2}(0,T;H_0) \cap W^{1,2}(0,T;V_0)$ such that $u_n(0) = u^0$, $\dot{u}_n(0) = u^1$, and, for almost all $t \in (0,T)$,

$$
\begin{aligned}
&(\ddot{u}_n, w - \dot{u}_n) + a_0(u_n, w - \dot{u}_n) + b_0(\dot{u}_n, w - \dot{u}_n) \\
&\quad + \phi_0(\gamma_1^n, \gamma_2^n, w) - \phi_0(\gamma_1^n, \gamma_2^n, \dot{u}_n) \geq \langle f_0, w - \dot{u}_n\rangle \quad \forall\, w \in V_0.
\end{aligned}
\tag{43}
$$

From (43), for $w = 0$, $w = 2\dot{u}_n$, and integrating over $(0,t)$ with $t \in (0,T)$, we derive

$$
\begin{aligned}
&\int_0^t (\ddot{u}_n, \dot{u}_n)\, ds + \int_0^t a_0(u_n, \dot{u}_n)\, ds + \int_0^t b_0(\dot{u}_n, \dot{u}_n)\, ds \\
&\quad + \int_0^t \phi_0(\gamma_1^n, \gamma_2^n, \dot{u}_n)\, ds = \int_0^t \langle f_0, \dot{u}_n\rangle\, ds,
\end{aligned}
$$

and so for almost every $t \in (0,T)$ we have

$$\frac{1}{2}|\dot u_n(t)|^2 + \frac{1}{2}a_0(u_n(t),u_n(t)) + \int_0^t b_0(\dot u_n,\dot u_n)\,ds$$
$$= -\int_0^t \phi_0(\gamma_1^n,\gamma_2^n,\dot u_n)\,ds + \int_0^t \langle f_0,\dot u_n\rangle\,ds + \frac{1}{2}|u^1|^2 + \frac{1}{2}a_0(u^0,u^0).$$

By the relations (14), (23) for $v_2=0$, and (13), we obtain

$$\frac{1}{2}|\dot u_n(t)|^2 + \frac{m_a}{2}\|u_n(t)\|^2 + m_b\int_0^t \|\dot u_n\|^2\,ds$$
$$\le \int_0^t k_2(\|\gamma_1^n\|_{L^2(\Xi)} + \|\gamma_2^n\|_{L^2(\Xi)})\|\dot u_n\|_U\,ds + \int_0^t \|f_0\|\|\dot u_n\|\,ds + \frac{1}{2}|u^1|^2 + \frac{M_a}{2}\|u^0\|^2.$$

Since the sequence $(\gamma_1^n,\gamma_2^n)_n$ is bounded in $(L^2(0,T;L^2(\Xi)))^2$, by Young's inequality it follows that there exists a positive constant C_1, depending only on a_0, b_0, f_0, u^0, u^1, the bound of $(\gamma_1^n,\gamma_2^n)_n$, k_2 and C_U, such that the following estimates hold:

$$\forall\, n\in\mathbb{N},\ |\dot u_n(t)|\le C_1,\ \|u_n(t)\|\le C_1 \text{ a.e. } t\in(0,T),\ \|\dot u_n\|_{L^2(0,T;V_0)}\le C_1. \tag{44}$$

Using (43) for $w=\dot u_n \pm \psi$ and (19), we see that for all $\psi\in L^2(0,T;U_0)$,

$$\int_0^T (\ddot u_n,\psi)\,ds + \int_0^T a_0(u_n,\psi)\,ds + \int_0^T b_0(\dot u_n,\psi)\,ds = \int_0^T \langle f_0,\psi\rangle\,ds.$$

This relation and the estimates (44) imply that there exists a positive constant C_2 having the same properties as C_1 and satisfying the estimate

$$\forall\, n\in\mathbb{N},\ \ \|\ddot u_n\|_{L^2(0,T;U_0')}\le C_2. \tag{45}$$

From (44), (45), it follows that there exist a subsequence $(u_{n_k})_k$ and u such that

$$\begin{aligned} &\dot u_{n_k} \rightharpoonup^* \dot u \text{ in } L^\infty(0,T;H_0),\quad u_{n_k}\rightharpoonup^* u \text{ in } L^\infty(0,T;V_0),\\ &\dot u_{n_k}\rightharpoonup \dot u \text{ in } L^2(0,T;V_0),\quad \ddot u_{n_k}\rightharpoonup \ddot u \text{ in } L^2(0,T;U_0'). \end{aligned} \tag{46}$$

According to Theorem 2.1 with

$$\begin{aligned} &\mathcal{F}=(\dot u_{n_k})_k,\ \ \hat X = H_0,\ \hat U = U', \hat Y = U_0',\ r=2,\\ &\mathcal{F}=(u_{n_k})_k,\ \ \hat X = V_0,\ \hat U = U,\ \hat Y = H_0,\ r=2,\\ &\mathcal{F}=(\dot u_{n_k})_k,\ \ \hat X = V_0,\ \hat U = U,\ \hat Y = U_0',\ p=2, \end{aligned}$$

we obtain

$$\dot{u}_{n_k} \to \dot{u} \text{ in } C([0,T];U'), u_{n_k} \to u \text{ in } C([0,T];U), \dot{u}_{n_k} \to \dot{u} \text{ in } L^2(0,T;U). \tag{47}$$

By Lemma 3.3, for all k, $l \in \mathbb{N}$ we have

$$\begin{aligned}\int_0^T \|\dot{u}_{n_k} - \dot{u}_{n_l}\|^2\,ds \le C_0 \int_0^T \{\phi_0(\gamma_1^{n_k}, \gamma_2^{n_k}, \dot{u}_{n_l}) - \phi_0(\gamma_1^{n_k}, \gamma_2^{n_k}, \dot{u}_{n_k})\\ +\phi_0(\gamma_1^{n_l}, \gamma_2^{n_l}, \dot{u}_{n_k}) - \phi_0(\gamma_1^{n_l}, \gamma_2^{n_l}, \dot{u}_{n_l})\}\,ds.\end{aligned} \tag{48}$$

and passing to limits by using (24), we find that $(\dot{u}_{n_k})_k$ is a Cauchy sequence in $L^2(0,T;V_0)$. Thus, $(\dot{u}_{n_k})_k$ is strongly convergent to $\dot{u}$ in this space and since

$$\text{for all } t \in [0,T], \quad u_{n_k}(t) = u^0 + \int_0^t \dot{u}_{n_k}(s)\,ds,$$

we deduce

$$u_{n_k} \to u \text{ in } C([0,T];V_0), u_{n_k} \to u \text{ in } W^{1,2}(0,T;V_0). \tag{49}$$

The limit u is the same for all the convergent subsequences, satisfying convergence properties similar to (47), corresponding to every sequence approximating (γ_1, γ_2), as can be readily seen by passing to limits in the following relation, obtained from (41) for $\gamma_{1,2} = \gamma_{1,2}^n$, $\delta_{1,2} = \delta_{1,2}^n$ and for all $n \in \mathbb{N}$:

$$\begin{aligned}&\int_0^T \|\dot{u}_{(\gamma_1^n,\gamma_2^n)} - \dot{u}_{(\delta_1^n,\delta_2^n)}\|^2\,ds\\ &\le C_0 \int_0^T \{\phi_0(\gamma_1^n, \gamma_2^n, \dot{u}_{(\delta_1^n,\delta_2^n)}) - \phi_0(\gamma_1^n, \gamma_2^n, \dot{u}_{(\gamma_1^n,\gamma_2^n)})\\ &+\phi_0(\delta_1^n, \delta_2^n, \dot{u}_{(\gamma_1^n,\gamma_2^n)}) - \phi_0(\delta_1^n, \delta_2^n, \dot{u}_{(\delta_1^n,\delta_2^n)})\}\,ds,\end{aligned} \tag{50}$$

where $(\delta_1^n,\, \delta_2^n)_n$ is an arbitrary sequence in $(W^{1,\infty}(0,T;L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2$ such that $\delta_1^n(0) = \delta_2^n(0) = 0\ \forall n \in \mathbb{N}$, and $(\delta_1^n, \delta_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0,T;L^2(\Xi)))^2$.

Now, for all $v \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0)$, we choose in (43) $w = \dot{u}_n + \frac{1}{k}(v - u_n)$, and so integrating over $(0,T)$ yields

$$\begin{aligned}&\int_0^T (\ddot{u}_n, v - u_n)\,dt + \int_0^T \{a_0(u_n, v - u_n) + b_0(\dot{u}_n, v - u_n)\}\,dt\\ &+ \int_0^T \{\phi_0(\gamma_1^n, \gamma_2^n, v - u_n + k\dot{u}_n) - \phi_0(\gamma_1^n, \gamma_2^n, k\dot{u}_n)\}\,dt\\ &\ge \int_0^T \langle f_0, v - u_n\rangle\,dt\end{aligned} \tag{51}$$

and integrating by parts the first term in (51) implies

$$
\begin{aligned}
&(\dot{u}_n(T), v(T) - u_n(T)) - (\hat{u}_1, v(0) - u^0) \\
&+ \int_0^T \{-(\dot{u}_n, \dot{v} - \dot{u}_n) + a_0(u_n, v - u_n) + b_0(\dot{u}_n, v - u_n)\}\, dt \qquad (52)\\
&+ \int_0^T \{\phi_0(\gamma_1^n, \gamma_2^n, v - u_n + k\dot{u}_n) - \phi_0(\gamma_1^n, \gamma_2^n, k\dot{u}_n)\}\, dt \geq \int_0^T \langle f_0, v - u_n\rangle\, dt.
\end{aligned}
$$

Using e.g. (47), (13) and (24), it is clear that we can pass to the limit in each term of (52) and so we obtain that $u = u_{(\gamma_1,\gamma_2)}$ is a solution of (42). $\square$

Let $\Phi : (L^2_+(\Xi_T))^2 \to 2^{(L^2_+(\Xi_T))^2} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$
\begin{aligned}
\Phi(\gamma_1, \gamma_2) = \Lambda_+(l(u_{(\gamma_1,\gamma_2)})) \times \Lambda_-(l(u_{(\gamma_1,\gamma_2)})) \\
\text{for all } (\gamma_1, \gamma_2) \in (L^2_+(\Xi_T))^2,
\end{aligned}
\qquad (53)
$$

where $u_{(\gamma_1,\gamma_2)}$ is the solution of the variational inequality (42) which corresponds to (γ_1, γ_2) by the procedure described in Theorem 3.2.

It is easily seen that if (λ_1, λ_2) is a fixed point of Φ, i.e. $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$, then $(u_{(\lambda_1,\lambda_2)}, \lambda_1, \lambda_2)$ is a solution of the Problem Q_2.

We shall consider a new problem, which consists in finding a fixed point of the set-valued mapping Φ, called also multivalued function or multifunction, which will provide a solution of Problem Q_1.

3.3 Existence of a fixed point

We shall prove the existence of a fixed point of the multifunction Φ by using a corollary of the Ky Fan's fixed point theorem [14], proved in [27] in the particular case of a reflexive Banach space.

Definition 3.1. *Let Y be a reflexive Banach space, D a weakly closed set in Y, and $F : D \to 2^Y \setminus \{\emptyset\}$ be a multivalued function. F is called sequentially weakly upper semicontinuous if $z_n \rightharpoonup z$, $y_n \in F(z_n)$ and $y_n \rightharpoonup y$ imply $y \in F(z)$.*

Proposition 3.1. *([27]) Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y, and $F : D \to 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $F(z)$ is convex for every $z \in D$. Then F has a fixed point.*

Note that since Y is a reflexive Banach space and D is convex, closed and bounded, there is no assumption that Y is separable, see [27, 4].

Theorem 3.3. *Assume that (1), (2), (13)-(21) and (25) hold. Then there exists $(\lambda_1, \lambda_2) \in (L^2_+(\Xi_T))^2$ such that $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$. For each fixed point (λ_1, λ_2) of the multifunction Φ, $(u_{(\lambda_1,\lambda_2)}, \lambda)$ with $\lambda = \lambda_1 - \lambda_2$ is a solution of the Problem Q_1.*

Proof. By Lemma 3.1, if $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$, then $(u_{(\lambda_1,\lambda_2)}, \lambda)$ is clearly a solution to the Problem Q_1.

We apply Proposition 3.1 to $Y = (L^2(0, T; L^2(\Xi)))^2$, $D = (L^2_+(\Xi_T))^2 \cap \left\{\zeta \in L^2(0, T; L^2(\Xi)); \|\zeta\|_{L^2(0,T;L^2(\Xi))} \leq R_0\right\}^2$ and $F = \Phi$.

The set $D \subset (L^2(0, T; L^2(\Xi)))^2$ is clearly convex, closed and bounded. Since for each $\zeta \in L^2(0, T; L^2(\Xi))$ the sets $\Lambda_+(\zeta)$ and $\Lambda_-(\zeta)$ are nonempty, convex, closed, and bounded by R_0, it follows that $\Phi(\gamma_1, \gamma_2)$ is a nonempty, convex and closed subset of D for every $(\gamma_1, \gamma_2) \in D$.

In order to prove that the multifunction Φ is sequentially weakly upper semicontinuous, let $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$, $(\gamma_1^n, \gamma_2^n) \in D$, $(\eta_1^n, \eta_2^n) \in \Phi(\gamma_1^n, \gamma_2^n)$ $\forall\, n \in \mathbb{N}$, $(\eta_1^n, \eta_2^n) \rightharpoonup (\eta_1, \eta_2)$ and let us verify that $(\eta_1, \eta_2) \in \Phi(\gamma_1, \gamma_2)$.

Using the Theorem 3.2 for each $n \in \mathbb{N}$, it follows that there exists a sequence $(\hat{\gamma}_1^n, \hat{\gamma}_2^n)_n$ in $(W^{1,\infty}(0, T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2$ such that $(\gamma_1^n, \gamma_2^n) - (\hat{\gamma}_1^n, \hat{\gamma}_2^n) \rightharpoonup 0$, $\hat{\gamma}_1^n(0) = \hat{\gamma}_2^n(0) = 0$ and

$$\|u_{(\hat{\gamma}_1^n,\hat{\gamma}_2^n)} - u_{(\gamma_1^n,\gamma_2^n)}\|_{W_0} \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \tag{54}$$

where $u_{(\hat{\gamma}_1^n,\hat{\gamma}_2^n)}$ is the solution of (37) corresponding to $(\hat{\gamma}_1^n, \hat{\gamma}_2^n)$, $u_{(\gamma_1^n,\gamma_2^n)}$ is the solution of (42) corresponding to (γ_1^n, γ_2^n) and to the procedure that enables to define $\Phi(\gamma_1^n, \gamma_2^n)$.

As $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0, T; L^2(\Xi)))^2$, Theorem 3.2 implies $u_{(\hat{\gamma}_1^n,\hat{\gamma}_2^n)} \to u_{(\gamma_1,\gamma_2)}$ in W_0, and by (54) and the triangle inequality, we obtain

$$u_{(\gamma_1^n,\gamma_2^n)} \to u_{(\gamma_1,\gamma_2)} \quad \text{in } W_0. \tag{55}$$

Now, by Lemma 3.1, the relation $(\eta_1^n, \eta_2^n) \in \Phi(\gamma_1^n, \gamma_2^n)$ is equivalent to

$$\eta_1^n - \eta_2^n \in \Lambda(l(u_{(\gamma_1^n,\gamma_2^n)})) \tag{56}$$

which may be rewritten as

$$\underline{\kappa} \circ l_n \leq \eta_1^n - \eta_2^n \leq \overline{\kappa} \circ l_n \quad \text{a.e. in } \Xi_T, \tag{57}$$

for all $n \in \mathbb{N}$, where $l_n := l(u_{(\gamma_1^n,\gamma_2^n)})$. The relations (57) are equivalent to

$$\int_\omega \underline{\kappa} \circ l_n \leq \int_\omega (\eta_1^n - \eta_2^n) \leq \int_\omega \overline{\kappa} \circ l_n, \tag{58}$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$.

Using (55), (15), the semi-continuity of $\underline{\kappa}$ and $\overline{\kappa}$, the relation (2), the convergence property $\int_\omega (\eta_1^n - \eta_2^n) \to \int_\omega (\eta_1 - \eta_2)$, and passing to limits according to Fatou's lemma (see also [27]), we obtain

$$\int_\omega \underline{\kappa} \circ l(u_{(\gamma_1,\gamma_2)}) \leq \int_\omega (\eta_1 - \eta_2) \leq \int_\omega \overline{\kappa} \circ l(u_{(\gamma_1,\gamma_2)}), \tag{59}$$

for every measurable subset $\omega \subset \Xi_T$, which implies $(\eta_1, \eta_2) \in \Phi(\gamma_1, \gamma_2)$. $\square$

4 Existence of a solution to the contact problem

Theorem 4.1. *Under the assumptions of Section 2 there exists a solution of the Problem P_v^1.*

Proof. We shall prove that there exists at least a solution $(\boldsymbol{u}, \lambda_+, \lambda_-)$ of the Problem P_v^2 which will provide a solution $(\boldsymbol{u}, \lambda)$ of the Problem P_v^1 with $\lambda = \lambda_+ - \lambda_-$.

We apply Theorem 3.3 to $U_0 = \boldsymbol{H}_0^1 = H_0^1(\Omega^1;\mathbb{R}^d) \times H_0^1(\Omega^2;\mathbb{R}^d)$, $V_0 = \boldsymbol{V}$, $U = \boldsymbol{H}^\iota$, $H_0 = \boldsymbol{H}$, $a_0 = a$, $b_0 = b$, $u^0 = \boldsymbol{u}_0$, $u^1 = \boldsymbol{u}_1$, $\phi_0 = \phi$, $f_0 = \boldsymbol{f}$ and to the mapping $l : \boldsymbol{V} \to L^2(\Xi)$ defined by $l(\boldsymbol{v}) = [v_N]$ $\forall \boldsymbol{v} \in \boldsymbol{V}$.

Since $\mathcal{A}_{ijkl}^\alpha$, $\mathcal{B}_{ijkl}^\alpha \in L^\infty(\Omega^\alpha)$ $\forall\, i,\, j,\, k,\, l = 1, \ldots, d$, $\alpha = 1, 2$, we obtain (13).

The condition meas$(\Gamma_U^\alpha) > 0$, the ellipticity properties of the coefficients $\mathcal{A}_{ijkl}^\alpha$, $\mathcal{B}_{ijkl}^\alpha$ and the Korn's inequality imply that there exist $m_a^\alpha, m_b^\alpha > 0$ such that

$$a^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{v}^\alpha) \geq m_a^\alpha \, \|\boldsymbol{v}^\alpha\|_{\boldsymbol{V}^\alpha}^2,\ \ b^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{v}^\alpha) \geq m_b^\alpha \, \|\boldsymbol{v}^\alpha\|_{\boldsymbol{V}^\alpha}^2 \ \ \forall\, \boldsymbol{v}^\alpha \in \boldsymbol{V}^\alpha,\ \ \alpha = 1, 2,$$

and we obtain

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq m_a \, \|\boldsymbol{v}\|^2,\ \ b(\boldsymbol{v}, \boldsymbol{v}) \geq m_b \, \|\boldsymbol{v}\|^2 \ \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{60}$$

where $m_a = \min(m_a^1, m_a^2)$, $m_b = \min(m_b^1, m_b^2)$.

Also, the properties (15)-(19), (21) and (25) can be easily verified.

Now, let C_{tr} be a positive constant such that $\|\boldsymbol{v}\|_{(L^2(\Xi))^d} \leq C_{tr} \|\boldsymbol{v}\|_{\boldsymbol{H}^\iota}$ for all $\boldsymbol{v} \in \boldsymbol{H}^{\iota}$. Using (11), the following estimates hold:

$$\forall\, \gamma_1,\, \gamma_2,\, \delta_1,\, \delta_2 \in L^2_+(\Xi),\, \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V},$$

$$|\phi(\gamma_1, \gamma_2, \boldsymbol{v}_1) - \phi(\gamma_1, \gamma_2, \boldsymbol{v}_2) + \phi(\delta_1, \delta_2, \boldsymbol{v}_2) - \phi(\delta_1, \delta_2, \boldsymbol{v}_1)|$$

$$= | - (\gamma_1 - \gamma_2, v_{1N})_{L^2(\Xi)} + \int_\Xi \mu\, (\gamma_1 + \gamma_2)\, |\boldsymbol{v}_{1T}|\, d\xi$$

$$+(\gamma_1 - \gamma_2, v_{2N})_{L^2(\Xi)} - \int_\Xi \mu\, (\gamma_1 + \gamma_2)\, |\boldsymbol{v}_{2T}|\, d\xi$$

$$-(\delta_1 - \delta_2, v_{2N})_{L^2(\Xi)} + \int_\Xi \mu\, (\delta_1 + \delta_2)\, |\boldsymbol{v}_{2T}|\, d\xi$$

$$+(\delta_1 - \delta_2, v_{1N})_{L^2(\Xi)} - \int_\Xi \mu\, (\delta_1 + \delta_2)\, |\boldsymbol{v}_{1T}|\, d\xi|$$

$$\leq |(\gamma_1 - \gamma_2 - \delta_1 + \delta_2, v_{1N} - v_{2N})_{L^2(\Xi)}|$$

$$+| \int_\Xi \mu\, (\gamma_1 + \gamma_2 - \delta_1 - \delta_2)\, (|\boldsymbol{v}_{1T}| - |\boldsymbol{v}_{2T}|)\, d\xi|$$

$$\leq (\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\, \|v_{1N} - v_{2N}\|_{L^2(\Xi)}$$

$$+\|\mu\|_{L^\infty(\Xi)}\, (\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\, \|\boldsymbol{v}_{1T} - \boldsymbol{v}_{2T}\|_{(L^2(\Xi))^d}$$

$$\leq (1 + \|\mu\|_{L^\infty(\Xi)})\, (\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{(L^2(\Xi))^d}$$

$$\leq C_{tr}(1 + \|\mu\|_{L^\infty(\Xi)})\, (\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)})\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{H}^{\iota}},$$

and so (20) is satisfied with $k_2 = C_{tr}(1 + \|\mu\|_{L^\infty(\Xi)})$. □

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